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Coupled Microstrip Disk Resonators

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***Abstract*—The coupling between microstrip disk resonators is investigated analytically and experimentally. The interaction between the printed disks is modeled by a gap capacitance, which is computed by solving the corresponding electrostatic problem. An integral equation is used to determine the nonsymmetric charge distribution on the disk resonators. Numerical results are presented for several cases. For a specific case the prediction of the theory is compared with the experiment.**

I. Introduction

THE GAP CAPACITANCE for microstrip printed circuits [1], [2] is investigated by several authors, where mostly linear edge shapes are treated. In this article the coupling between printed disk resonators is considered.

The geometry of the problem is defined in Fig. 1. The coupled disk resonators are printed on a grounded dielectric substrate. The substrate thickness is H with a relative dielectric constant ϵ_r. Also a second perfect conductor-ground plane is assumed at $z=B$. The coupling between the two resonators is assumed to be mainly due to the fringing effects of the electric fields; and, as a result, the coupling between the two disks can be modeled by a gap capacitance C_g. In Section II a method for computing C_g is developed. The method is based on using the cylindrical coordinates in conjunction with Galerkin technique. The

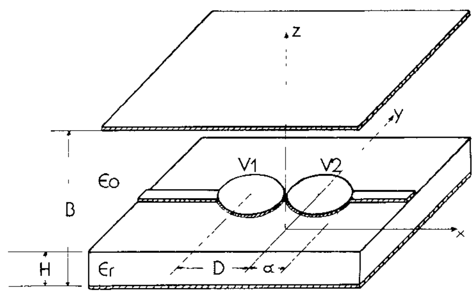

Fig. 1. Coupled resonator geometry.

resonance frequencies for disk resonators can be computed by assuming infinite magnetic conductivity resonators walls [3]. The behavior of coupled resonators can be predicted by considering an equivalent circuit around each resonance frequency.

Assuming the disks to be raised at $V_1=1/2$ and $V_2=-1/2$ V the total charge on each disk will be

$$Q(D)=C_g(D)+(1/2)C \quad (1)$$

where C is the self capacitance of each disk. For very large D values

$$\lim_{D\to+\infty} Q(D)=(1/2)C$$

so the gap capacitance will be

$$C_g(D)=Q(D)-\lim_{D\to+\infty} Q(D). \quad (2)$$

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II. Evaluation of $C_g(D)$ Gap Capacitance

An integral equation method is used to handle the electrostatic problem for coupled disk resonators. In order to decrease the complexity due to the boundary conditions, first the Green's function $G(\mathbf{r},\mathbf{r}')$ of the problem in the absence of the printed disk conductors is determined. The Green's function is defined as the response of the electric field to a unit charge excitation.

In terms of the cylindrical harmonic functions $G(\mathbf{r},\mathbf{r}')$ (see Appendix I) is given as

$$G(\rho,\phi/\rho',\phi') = \int_0^{+\infty} dt \sum_{m=-\infty}^{+\infty} J_m(t\rho)J_m(t\rho')e^{im(\phi-\phi')}W(t,B,H,\epsilon r) \quad (3)$$

where $W(t,B,H,\epsilon r) = \{(\coth(t(B-H)) + \epsilon r \coth(tH)) \cdot 2\pi\epsilon_0\}^{-1}$ and (ρ,ϕ), (ρ',ϕ') are the cylindrical coordinates on the dielectric substrate (in (3) the source and observation points are assumed to be at $z=z'=H$).

Given the disk voltages $V_1=-V_2=1/2$ the charge distributions $q_1(\rho_1,\phi_1)$, $q_2(\rho_2,\phi_2)$ on the disks satisfies the symmetry relations

$$q_1(\rho_1,\phi_1)=q_1(\rho_1,-\phi_1)=-q_2(\rho_2,\pi-\phi_2) \quad (4)$$

where (ρ_1,ϕ_1), (ρ_2,ϕ_2) are the local coordinates of the two disks (see Fig. 1). The integral equation for the charge distribution will be

$$\int_{D1} G(\rho_1,\phi_1/\rho_1',\phi_1')q_1(\rho_1',\phi_1')\rho_1'd\rho_1'd\phi_1' + \int_{D2} G(\rho_2,\phi_2/\rho_2',\phi_2')q_2(\rho_2',\phi_2')\rho_2'd\rho_2'd\phi_2' = 1/2 \quad (5)$$

where $D1$, $D2$ are for the areas of the two disks. The point $(\rho_1,\phi_1)=(\rho_2,\phi_2)$ is assumed to be on $D1$. Applying the translational addition theorem for cylindrical waves [4]

$$J_m(t\rho_2)e^{im\phi_2} = \sum_{k=-\infty}^{+\infty} J_k(tD)J_{m+k}(t\rho_1)e^{i(m+k)\phi_1}(-1)^k \quad (6)$$

and using the symmetry relation (4), (6) can be written as

$$\int_{D1} \rho'd\rho'd\phi'\{G(\rho,\phi/\rho',\phi')q_1(\rho',\phi') - G'(\rho,\phi/\rho',\phi')q_1(\rho',\pi-\phi')\} = 1/2 \quad (7)$$

where $(\rho,\phi)\epsilon\, D1$ and

$$G'(\rho,\phi/\rho',\phi') = \int_0^{+\infty} dt \sum_{m=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} J_k(tD)J_{m+k}(t\rho)J_m(t\rho') \cdot W(t,B,H,\epsilon_r)e^{im(\phi-\phi')}e^{ik\phi}(-1)^k.$$

Defining the Fourier coefficients

$$(1/2\pi)\int_0^{+\infty} e^{-im\phi'}\begin{pmatrix} q_1(\rho',\phi') \\ q_1(\rho',\pi-\phi') \end{pmatrix} d\phi' = \begin{pmatrix} 1 \\ (-1)^m \end{pmatrix} q_m(\rho') \quad (8)$$

and multiplying (7) by $e^{-in\phi}$, after a Fourier integration, the following equation is obtained:

$$\int_0^{\alpha} \rho'd\rho' \sum_{m=-\infty}^{+\infty} T_{nm}(\rho,\rho')q_m(\rho') = \delta_{n0}(1/2) \quad (9)$$

δ_{n0} being the Kronecker delta and

$$T_{nm}(\rho,\rho') = \int_0^{+\infty} dt J_n(t\rho)J_m(t\rho') \cdot (\delta_{nm} - J_{n-m}(tD)(-1)^n)W(t,B,H,\epsilon_r). \quad (10)$$

In order to determine the unknown coefficients $\{q_m(\rho')\}$ a linearly independent set of functions is

$$q_m(\rho') = \sum_{r=0}^{N} I_m(s_r\rho')C_r(m) \quad (11)$$

where $I_m(x), m=0,\pm 1,\pm 2,\cdots$ are the modified Bessel functions. The real numbers $s_0,s_1,s_2,\cdots$ are pivots and $C_r(m)$ are unknown coefficients. The choice of modified Bessel functions as a basis set is very appropriate since these are monotonically increasing functions and as a result are natural to the charge distribution on the disk.

A similar basis function has been used successfully for microstrip line problems [5]. In order to apply the Galerkin method, (11) is substituted in (9), then multiplying both sides by $I_n(s_p\rho)$ (with $\{s_p\}=\{s_r\}$), after an integration the following set of equations are obtained:

$$\sum_{m=-M}^{+M}\sum_{r=0}^{N} S_{nm}(s_p,s_r)C_r(m) = \delta_{n0}\alpha^2(I_1(s_p\alpha))/(2s_p\alpha), \quad \text{for } \begin{matrix} n=-M,\cdots,M \\ p=0,\cdots,N \end{matrix} \quad (12)$$

where the Fourier summations are truncated to finite summations (the expression for $S_{nm}(s_p,s_r)$ is given in Appendix II). Equation (12) constitutes a set of linear simultaneous equations and can be solved numerically. Assuming the $\det\{S_{nm}(s_p,s_r)\}\neq 0$, then the vector $C_r(m)$ can be determined and the total charge on each disk approximately will be

$$Q = \int_{D1} \rho'd\rho'd\phi' q_1(\rho',\phi') = 2\pi\alpha^2 \sum_{r=0}^{N} C_r(0)(I_1(s_r\alpha))/s_r\alpha. \quad (13)$$

It is well known that the values of Q obtained by solving (12) and (13) are variational since a Galerkin procedure [6], [7] is employed. This ensures the fast convergence and the numerical stability of the solution as it is shown below. Note that although the initial integral (5) has a singular kernel, the application of Galerkin technique removes this singularity after the integrations over the variables φ,φ' and ρ,ρ' [8].

III. Numerical Computations

In order to check the numerical accuracy of the solution convergence tests have been performed. In Table I a convergence pattern for the computed Q values is given. In all of the performed numerical tests a perfect convergence pattern is observed. Even for very strong coupled disks ($D\simeq 2\alpha$) $M=2, N=2$ the method was found to give convergent results. Numerical computations have been

TABLE I
CONVERGENCE PATTERN OF Q FOR $\alpha/H=1.428$, $D/H=2.94$, $B/H=4.3$, and $\epsilon_r=3.09$

i \ M	1	2	3
2	0.688(-12)	0.690(-12)	0.693(-12)
3	0.69[illegible](-12)	0.685(-12)	0.692(-12)
4	0.690(-12)	0.695(-12)	0.694(-12)

TABLE II.
RESULTS FOR THE COUPLING COEFFICIENT $k=C_g/C$ AND SELF CAPACITANCE C FOR VARIOUS ϵ_r, α/H AND $S/H=(D-2\alpha)/H$ INTERDISK SPACINGS ($B/H=6$).

α/H=2.0	k	k	k
S/H \ εr	2.3	3.09	9.9
0.05	4.0(-2)	4.4(-2)	3.7(-2)
0.10	3.0(-2)	4.3(-2)	3.0(-2)
0.30	1.5(-2)	2.2(-2)	2.0(-2)
0.60	1.3(-2)	1.7(-2)	1.6(-2)
C (pF/mm)	0.470	0.500	1.060

α/H=3.0	k	k	k
S/H \ εr	2.3	3.09	9.9
0.05	4.9(-2)	6.1(-2)	2.4(-2)
0.10	3.9(-2)	5.4(-2)	1.2(-2)
0.30	2.9(-2)	4.1(-2)	0.9(-2)
0.60	1.8(-2)	3.1(-2)	0.5(-2)
C (pF/mm)	0.899	1.120	3.260

α/H=10.0	k	k	k
S/H \ εr	2.3	3.09	9.9
0.05	1.9(-3)	10.0(-3)	10.0(-3)
0.10	0.6(-3)	9.0(-3)	6.0(-3)
0.30	0.5(-3)	8.0(-3)	3.3(-3)
0.60	0.3(-3)	7.0(-3)	3.0(-3)
C (pF/mm)	7.730	9.500	30.00

carried out in several cases. In Table II the coupling coefficient $k=C_g/C$ is given for several α/H, ϵ_r and D/H values. Results reveal that the gap capacitance C_g is a decreasing function of the dielectric constant ϵ_r.

IV. EXPERIMENTAL COMPARISON

Two equi-radii disks of $\alpha=5.8$ mm and $D=11.68$ mm (see Fig. 1) have been printed on a polyplate ($\epsilon_r=2.3$) microstrip substrate with $H=0.79$ mm and $B\gg H$. The reflection coefficient ρ for a 50-Ω input impedance is measured from the input of the one disk while the second was left open as shown in Fig. 2. A HP network analyzer is used to measure the ρ in amplitude and phase. For the dominant resonant mode an equivalent circuit is used to deduce the input impedance as

$$Z_i=\frac{r}{1+jQ(1-f_0^2/f^2)}\cdot\frac{1+jQ(k+1)(1-f_1^2/f^2)}{1+jQ(2k+1)(1-f_2^2/f^2)}$$

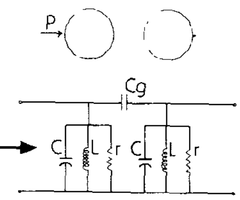

Fig. 2. Equivalent circuit for the coupled resonators around the dominant mode.

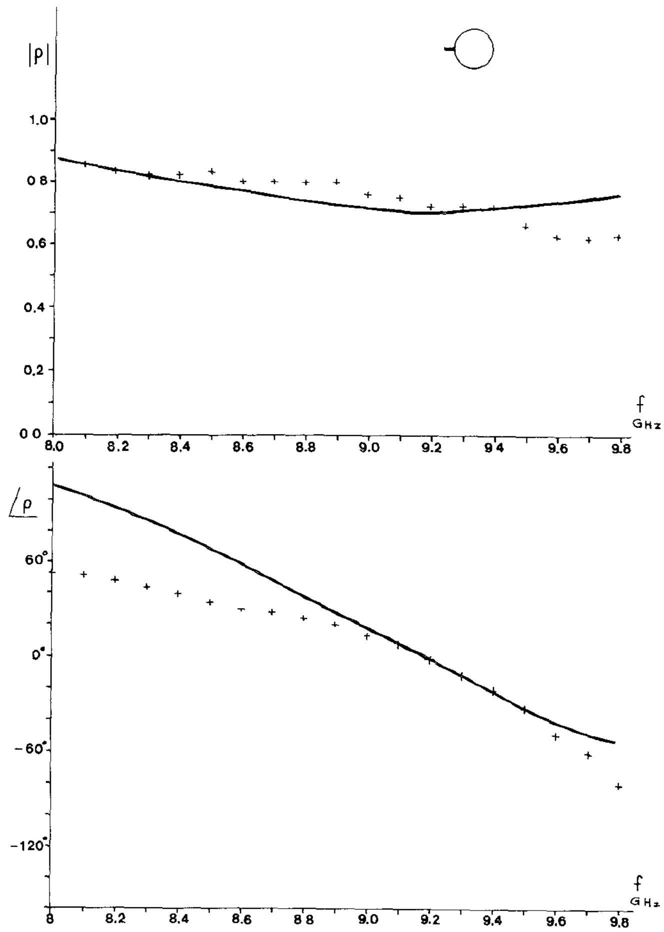

Fig. 3. Theoretical (——) and experimental (+ + +) reflection coefficient ρ versus the frequency for a single resonator $\alpha/H=7.34$, $B\gg H$, and $\epsilon_r=2.3$.

where $f_1=f_0(1+k)^{-1/2}$, $f_2=f_0(1+2k)^{-1/2}$, and f_0, Q, r are for the resonance frequency, quality factor, and shunt resistance of the single disk. In order to determine the f_0, Q, r, a single disk of the same dimensions and substrate is measured. In Fig. 3 the reflection coefficient ρ for a single resonator is presented: it is found that $f_0\simeq9.20$ GHz, $Q\simeq24$, and $r\simeq300$ Ω. The measured f_0 is in agreement with the quasi-static prediction methods [3]. The coupling coefficient for the nearby disks is computed to be $k=0.05$. In Fig. 4 the comparison between the measured and computed values of ρ is given. A reasonable agreement is observed around the f_0 resonance where the equivalent circuit model is valid. Since $kQ=1.2$, the well-known strong coupling effects are observed.

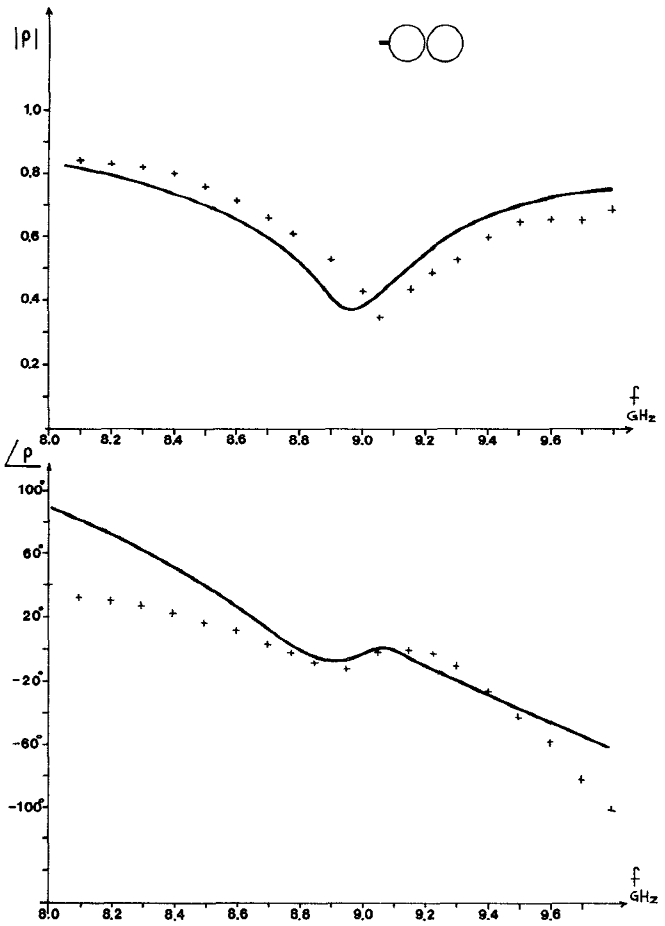

Fig. 4. Comparison of theoretical (——) and experimental (+ + +) reflection coefficients for two coupled disks as defined in Fig. 3 and $(D-2\alpha)/H = 0.1$.

V. Conclusion

The coupling between two microstrip disk resonators is investigated theoretically and experimentally. A Galerkin procedure is developed and used for the computation of gap capacitances for strongly coupled disk resonators. The numerical method which is used is very stable and only few expansion terms are required to ensure convergence. The comparison of theory with an experimental setup shows a reasonable agreement, which indicates that the developed numerical technique can be used in other printed circuit design problems.

Appendix I

Evaluation of Green's Function

Assume a $q = \delta(x)\delta(y)$ unit charge excitation at $x = y = 0$, $z = H$ of Fig. 1, when the printed disks are absent.

The electrostatic potential G_1 (which satisfies the Laplace equation) for $z < H$ in terms of cylindrical harmonic functions can be written as

$$G_1 = \int_0^{+\infty} tJ_0(t\rho)\sinh(tz)g_1 dt \qquad \text{(A.1.1)}$$

and for $B > z > H$

$$G_2 = \int_0^{+\infty} tJ_0(t\rho)\sinh(t(B-z))g_2 dt. \qquad \text{(A.1.2)}$$

Using the boundary conditions at the dielectric-air interface

$$G_1 = G_2$$

$$\epsilon_0 \partial G_2/\partial z - \epsilon r \cdot \epsilon_0 \partial G_1/\partial z = (\delta(\rho))/\rho = (1/2\pi)\int_0^{+\infty} J_0(t\rho)dt$$

for $z = H$ and (A.1.1) and (A.1.2) the unknown coefficients g_1, g_2 are determined. For $z = H$ the Green's function is given as

$$G = \int_0^{+\infty} dt W(t, B, H, \epsilon r) J_0(t\rho). \qquad \text{(A.1.3)}$$

If the charge is located at (ρ', ϕ'), by using the translational addition theorem for cylindrical wave functions [4] the Green's function given in (3) is obtained.

Appendix II

Evaluation of S_{nm} Matrix Elements

Following a straightforward algebra the S_{nm} matrix elements are obtained as

$$S_{nm} = \int_0^{+\infty} dt(\delta_{nm} - J_{n-m}(tD)(-1)^n) W(t, B, H, \epsilon r) \cdot \gamma_n(s_p, t, \alpha)\gamma_m(s_r, t, \alpha) \qquad \text{(A.2.1)}$$

where

$$\gamma_n(s_p, t, \alpha) = \{\alpha/(s_p^2 + t^2)\} \cdot \{tJ_{n+1}(t\alpha)I_n(s_p\alpha) + s_p J_n(t\alpha)I_{n+1}(s_p\alpha)\}.$$

The integration in (A.2.1) is performed numerically by using a Romemberg Check-Newton Cotes- numerical algorithm [9]. Note that for $t \to +\infty$ the integrand converges as fast as $0(t^{-4})$.

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